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NUMERICAL AND ASYMPTOTIC STUDY OF THE TWO-DIMENSIONAL PROBLEM OF THE HYDROELASTIC BEHAVIOR OF A FLOATING PLATE IN WAVES

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The problem of the behavior of a floating elastic plate in waves is solved numerically. The normal mode method is used. For a fluid of finite depth, the hydrodynamic coefficients are obtained in explicit form. Numerical results are compared with experimental data for the stress distribution in the plate and also with numerical results of other authors. The results are in good agreement for not very short waves. For incident waves whose wavelength is comparable with the length of the plate, a long-wave approximation of the solution is proposed. Within the framework of this approximation, the solution is given in analytical form.

We consider the two-dimensional linear problem of a floating elastic plate in waves. The length of the isotropic plate is $2a$ and its thickness is h, and $h/a \ll 1$. A fluid layer of finite depth H is bounded from below by an impermeable bottom. The upper boundary of the fluid consists of a free surface and the surface of the floating plate. Vibrations of the plate are generated by a plane surface wave of small amplitude. The problem is to determine the deflection of the plate, the stress distribution in it, and the transmission and reflection coefficients of the incident wave as functions of the wavelength and the parameters of the plate.

This problem has been studied experimentally and numerically [1, 2]. In the experiments, plates with dimensions 10.0×0.5 m [1] and 50×5 m [2] were placed in a narrow tank. In this case, the fluid flow induced by interaction between the incident plane wave and the floating plate can be approximately considered two-dimensional (which does not change across the plate) and the plate can be treated as an Euler free–free beam. In the numerical calculations, the normal mode method [1] and the method of a boundary integral equation [2] were used. In the latter method, the original linear problem is reduced using Green's function to a two-dimensional (one-dimensional in the plane case) Fredholm integral equation for the distribution of the hydrodynamic pressure on the plate. This integral equation is solved numerically. In the normal mode method [1], the beam deflection is represented as a superposition of its free vibration forms in air. The interaction between the beam and the fluid is described by an added-mass matrix, and the added masses are calculated for each vibration mode of the beam. This matrix plays a key role in the normal mode method. If the matrix is known, the problem reduces to simple calculations using explicit formulas. In [1], the elements of the matrix are determined by solving the hydrodynamic part of the problem using the method of flow domain decomposition. The calculation results in [1, 2] are in good agreement with experimental data for long incident waves. There is, however, a considerable discrepancy for short waves.

The method of flow domain decomposition was also used in [3, 4]. A disadvantage of this method is that it cannot be extended to the case of an infinitely deep fluid. At the same time, this method is effective for studying the behavior of a floating plate whose draft is comparable with the depth of the fluid layer.

The aim of the present paper is a comparison of the numerical results obtained by the direct normal mode method, in which the matrix elements are calculated explicitly, with the experimental data of [1, 2],

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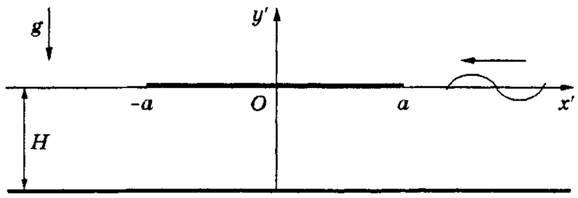

Fig. 1

numerical calculations [1–4], and asymptotic formulas of the long-wave approximation.

At the present time, there are projects of very large floating structures (airports and islands), most of which are based on the concept of a floating plate. This structure is easier to build from standard elements. It is stable and does not require powerful holding devices [5]. The giant dimensions of the real structure make full-scale experimental studies of its behavior difficult. Results of the laboratory experiments [1, 2] can be applied to the real structures, but one cannot guarantee that all essential similarity criteria are satisfied. In this situation, it is of special importance to study the hydroelastic behavior of floating plates by both numerical and analytical methods and to compare the results with the data of laboratory experiments. Such studies help to refine the models and their range of applicability.

Formulation of the Problem. We study the hydroelastic behavior of a floating plate (Fig. 1) within the framework of the linear theory. The draft of the plate d is assumed to be small in comparison with its length $2a$ and the depth of the fluid H. Time-periodic vibrations of the plate are generated by a surface wave of small amplitude A that is incident on the plate from right. The middle of the plate is taken as the origin of the Cartesian coordinates $x'Oy'$. Here and below, primes refer to dimensional variables. The fluid layer ($-H < y' < 0$) is bounded from below by a horizontal rigid bottom ($y' = -H$). The segments of the upper boundary ($y' = 0$) of the fluid layer $x' < -a$ and $x' > a$ correspond to the free surface and the segment $-a < x' < a$ corresponds to the floating plate. The plate is treated as an Euler free–free beam. We assume that the fluid is ideal, heavy, and incompressible, and the flow is two-dimensional and irrotational. In the linear theory, the fluid flow is described by the velocity potential $\varphi'(x', y', t')$, and the vibrations of the plate are described by the plate deflection $w'(x', t')$, where t' is the time. Below we use the following dimensionless variables: $x' = ax$, $y' = ay$, $t' = t/\omega$, $\varphi' = A\omega a\varphi$, $w' = Aw$, $p' = \rho gAp$, and $\eta' = A\eta$. Here ω is the frequency of the incident wave, $p(x, y, t)$ is the hydrodynamic pressure, g is the acceleration of gravity, ρ is the fluid density, and the equation $y = \eta(x, t)$, where $|x| > 1$, describes the evolution of the free surface. We note that, in dimensionless variables, the frequency and amplitude of the incident wave are equal to unity.

In dimensionless variables, the equations of motion and the boundary conditions take the form

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad (-\infty < x < +\infty, -H_0 < y < 0), \qquad \varphi_y = 0 \qquad (y = -H_0),$$

$$\varphi_y = \eta_t, \qquad \gamma\varphi_t + \eta = 0 \qquad (y = 0, \quad |x| > 1), \tag{1}$$

$$\varphi_y = w_t, \qquad p(x, 0, t) = -\gamma\varphi_t - w(x, t) \qquad (y = 0, \quad |x| < 1),$$

$$\alpha w_{tt} + \beta w_{xxxx} = p(x, 0, t) \qquad (|x| < 1), \qquad w_{xx} = w_{xxx} = 0 \qquad (x = \pm 1).$$

Here $\gamma = a\omega^2/g$, $H_0 = H/a$, $\alpha = \gamma(d/a)$, $\beta = EJ/(\rho g a^4)$, E is Young's modulus for the material of the beam, $J = h^3/12$, and h is the thickness of the plate. In writing the formula for the coefficient α, we used the equation of balance of forces in the case where the plate is floating on calm water: $m_b g = \rho g d$ (m_b is the mass of the beam per unit length).

We seek a solution of the problem (1) subject to the following conditions on the behavior of the free surface as $x \to \pm\infty$:

$$\eta(x, t) \sim \cos(kx + t) + A^{(+)} \cos(kx - t + \delta^{(+)}) \qquad (x \to +\infty),$$

$$\eta(x, t) \sim A^{(-)} \cos(kx + t + \delta^{(-)}) \qquad (x \to -\infty), \tag{2}$$

where $A^{(+)}$ and $A^{(-)}$ are the amplitudes of the reflected and transmitted waves divided by the amplitude of the incident wave, $\delta^{(+)}$ and $\delta^{(-)}$ are the corresponding phase shifts, and k is a dimensionless wavenumber that is a positive solution of the equation $k \tanh kH_0 = \gamma$. The wavelength of the incident wave λ in dimensional variables is given by the formula $\lambda = 2\pi a/k$. The quantities $A^{(+)}$, $A^{(-)}$, $\delta^{(+)}$, and $\delta^{(-)}$ are not known in advance and must be determined along with the flow parameters and the plate deflection $w(x,t)$. The relative extensions of elements of the beam $\varepsilon(x,t)$ are defined by the formula

$$\varepsilon(x,t) = \varepsilon_s w_{xx}(x,t), \qquad \varepsilon_s = Ah/(2a^2). \tag{3}$$

We note that initial data are absent in the formulation of the problem (1), (2). It is assumed that at long times the flow is time-periodic and independent of the distinctive features of the initial data. The velocity potential $\varphi(x,y,t)$ corresponding to the developed wave motion of the fluid, the beam deflection $w(x,t)$, and the pressure $p(x,y,t)$ are sought in the form

$$\begin{gathered} \varphi(x,y,t) = \varphi_{\rm i}(x,y,t) + \mathrm{Re}\,[i\exp(it)\Phi(x,y)], \quad w(x,t) = \mathrm{Re}\,[\exp(it)W(x)], \\ p(x,y,t) = \mathrm{Re}\,[\exp(it)P(x,y)], \quad \varphi_{\rm i}(x,y,t) = -\frac{1}{\gamma}\frac{\cosh[k(y+H_0)]}{\cosh(kH_0)}\sin(kx+t), \end{gathered} \tag{4}$$

where $\varphi_{\rm i}(x,y,t)$ is the velocity potential for the incident wave in the case without a plate. The new unknown functions $\Phi(x,y)$, $W(x)$, and $P(x,y)$ are complex-valued. Substituting Eqs. (4) into (1), we find that

$$\Phi_{xx} + \Phi_{yy} = 0 \qquad (-\infty < x < +\infty, \quad -H_0 < y < 0); \tag{5}$$

$$\Phi_y = 0 \qquad (y = -H_0); \tag{6}$$

$$\Phi_y = \gamma\Phi \qquad (y = 0, \quad |x| > 1); \tag{7}$$

$$\Phi_y = W(x) - \exp(ikx) \qquad (y = 0, \quad |x| < 1); \tag{8}$$

$$\beta\frac{d^4W}{dx^4} + (1-\alpha)W = \gamma\Phi(x,0) + \exp(ikx) \qquad (|x| < 1); \tag{9}$$

$$\frac{d^2W}{dx^2} = \frac{d^3W}{dx^3} = 0 \qquad (x = \pm 1); \tag{10}$$

$$P(x,0) = \gamma\Phi(x,0) - W(x) + \exp(ikx) \qquad (|x| < 1). \tag{11}$$

The radiation conditions (2) written in terms of the new variables have the forms

$$\Phi(x,0) \sim B^{(+)}\exp(-ikx) \quad (x \to +\infty), \qquad \Phi(x,0) \sim B^{(-)}\exp(ikx) \quad (x \to -\infty), \tag{12}$$

where the coefficients $B^{(+)}$ and $B^{(-)}$ need to be determined, and $A^{(\pm)} = \gamma B^{(\pm)}$, $\delta^{(+)} = -\arg B^{(+)}$, and $\delta^{(-)} = \arg B^{(-)}$.

The problem is to determine the reflection $A^{(+)}$ and transmission $A^{(-)}$ coefficients, the amplitude of the plate deflection $|W(x)|$, and the amplitude of the relative extensions $E(x) = \max_t |\varepsilon(x,t)| = \varepsilon_s|W_{xx}(x)|$ using the known parameters of the incident wave and the beam.

Normal Mode Method. According to the normal mode method, the deflection and the distribution of the hydrodynamic pressure along the plate are presented in the form of expansions

$$W(x) = \sum_{n=1}^{\infty} A_n\psi_n(x), \qquad P(x,0) = \sum_{n=1}^{\infty} P_n\psi_n(x) \qquad (|x| < 1), \tag{13}$$

where A_n and P_n are unknown complex coefficients and $\psi_n(x)$ are nontrivial real solutions of the spectral problem

$$\psi_n^{\rm IV} = \lambda_n^4\psi_n \quad (-1 < x < 1), \quad \psi_n''(\pm 1) = \psi_n'''(\pm 1) = 0, \quad \int_{-1}^{1}\psi_n(x)\psi_m(x)\,dx = \delta_{nm}. \tag{14}$$

Here λ_n are the corresponding eigenvalues, $\delta_{nm} = 0$ for $n \neq m$, and $\delta_{nm} = 1$ for $n = m$. Eigenfunctions with even numbers are even functions of x, and eigenfunctions with odd numbers are odd functions of x:

$$\psi_1(x) = \sqrt{3/2}\,x, \qquad \psi_{2n+1}(x) = D_{2n+1}(\sin\lambda_{2n+1}x + S_{2n+1}\sinh(\lambda_{2n+1}x)),$$

$$\psi_2(x) = 1/\sqrt{2}, \qquad \psi_{2n}(x) = D_{2n}(\cos\lambda_{2n}x + S_{2n}\cosh(\lambda_{2n}x)),$$

$$D_{2n+1} = 1\Big/\sqrt{1 - \cos^2\lambda_{2n+1}/\cosh^2\lambda_{2n+1}}, \qquad S_{2n+1} = \cos\lambda_{2n+1}/\cosh\lambda_{2n+1},$$

$$D_{2n} = 1\Big/\sqrt{1 + \cos^2\lambda_{2n}/\cosh^2\lambda_{2n}}, \qquad S_{2n} = \cos\lambda_{2n}/\cosh\lambda_{2n}.$$

The eigenvalues λ_{2n} ($n \geqslant 2$) and λ_{2n+1} ($n \geqslant 1$) are obtained from the dispersion relations $\tan\lambda_{2n+1} = \tanh\lambda_{2n+1}$ and $\tan\lambda_{2n} = -\tanh\lambda_{2n}$. In addition, $\lambda_1 = \lambda_2 = 0$.

Equations (9), (11), and (14) allow us to find the relation between the coefficients A_n and P_n:

$$P_n = -g_n A_n, \qquad g_n = \alpha - \beta\lambda_n^4. \tag{15}$$

Equations (7), (8), and (11) show that $\Phi_y - \gamma\Phi = -P(x,0)$ at $y = 0$. From this condition, the velocity potential $\Phi(x,y)$ is written as

$$\Phi(x,y) = -\sum_{n=1}^{\infty} P_n \Phi^{(n)}(x,y), \tag{16}$$

where the functions $\Phi^{(n)}(x,y)$ satisfy Eqs. (5) and (6), the boundary condition on the upper boundary of the fluid layer

$$\Phi_y^{(n)} - \gamma\Phi^{(n)} = \psi_n(x)H(1 - x^2) \qquad (y = 0),$$

and the radiation conditions

$$\Phi^{(n)}(x,0) \sim b_n^{(+)}\exp(-ikx) \quad (x \to +\infty), \qquad \Phi^{(n)}(x,0) \sim b_n^{(-)}\exp(ikx) \quad (x \to -\infty).$$

If the functions $\Phi^{(n)}(x,y)$ and the numbers $b_n^{(+)}$ and $b_n^{(-)}$ are found, the formula

$$B^{(\pm)} = -\sum_{n=1}^{\infty} P_n b_n^{(\pm)}$$

and the representation

$$\Phi^{(n)}(x,0) = \sum_{m=1}^{\infty} C_{nm}\psi_m(x) \qquad (|x| < 1) \tag{17}$$

with complex coefficients C_{nm} are valid. We note that these coefficients do not depend on the parameters of the beam.

Substituting (16), (17), and (13) into (11) and using (15), we obtain the following system of algebraic equations for the coefficients P_n:

$$\Big(1 - \frac{1}{g_n}\Big)P_n + \gamma\sum_{m=1}^{\infty} P_m C_{mn} = \int_{-1}^{1} \psi_n(x)\exp(ikx)\,dx. \tag{18}$$

System (18) is solved numerically by the reduction method if the hydrodynamic coefficients C_{mn} are known. Then, the coefficients A_n in the expansion of the deflection in terms of eigenfunctions (13) are determined from formulas (15).

Hydrodynamic Coefficients. The functions $\Phi^{(n)}(x,y)$ are given by

$$\Phi^{(n)}(x,y) = \frac{1}{2\pi}\int_L \psi_n^F(\xi)\,\frac{\exp(i\xi x)\,d\xi}{G(\xi) - \gamma}, \tag{19}$$

where $\psi_n^F(\xi) = \int\limits_{-1}^{1} \psi_n(x_0) \exp(-ix_0\xi)\, dx_0$, $G(\xi) = \xi \tanh \xi H_0$, and $G(k) = \gamma$. In (19), the integration path L in the plane of the complex variable ξ goes along the real axis and passes below the pole at the point $\xi = -k$ and above the pole at point $\xi = k$. With this choice of the integration contour, the radiation conditions are automatically satisfied.

Analysis of the behavior of the integral (19) as $x \to \pm\infty$ yields $b_n^{(\pm)} = -i\psi_n^F(\mp k)/G'(k)$, and, therefore,

$$B^{(\pm)} = \frac{i}{G'(k)} \sum_{n=1}^{\infty} P_n \psi_n^F(\mp k). \tag{20}$$

The hydrodynamic coefficients are determined from the formula

$$C_{nm} = \int\limits_{-1}^{1} \Phi^{(n)}(x,0)\psi_m(x)\, dx,$$

which, in view of (19), gives

$$C_{nm} = C_{nm}^R - iC_{nm}^I, \qquad C_{nm}^R = \frac{1}{2\pi} \text{v.p.} \int\limits_{-\infty}^{\infty} \frac{\psi_n^F(\xi)\psi_m^F(\xi)}{G(\xi) - \gamma}\, d\xi, \tag{21}$$

$$C_{nm}^I = \frac{1}{2G'(k)} [\psi_n^F(-k)\psi_m^F(k) + \psi_n^F(k)\psi_m^F(-k)].$$

One can see that $C_{nm} = C_{mn}$ if n and $m \geqslant 1$ and $C_{nm} = 0$ if $n+m$ an odd number. In the case of the fluid of finite depth, by deformation the integration path in (21), the integral understood in the sense of the Cauchy principle value can be represented in the form of a series that is convenient for numerical calculations.

Numerical Results. In numerical calculation of system (18), we retain 60 even and 60 odd modes. A comparison of the numerical results for the dimensionless amplitude of bending stresses obtained using the proposed model with results of the calculations in [4] for the experimental conditions of [1], is presented in Fig. 2. The curves obtained by the normal mode method are shown by the solid curve and the results of [4] are shown by the dashed curve. For convenience of comparison with experimental data, the results are presented in the same form as in [1]. Here $\bar{M} = |M(x)|/(2A\rho a\, dg) = (\beta a/d)|W''(x)|$, where $|M(x)|$ is the amplitude of bending stresses, and T is the period of the incident wave. One can see that for sufficiently long waves (the period $T = 1.429$ sec corresponds to a wavelength of 3.1 m, and the period $T = 2.875$ sec corresponds to a wavelength of 8.6 m), the numerical results obtained by the two different methods practically coincide. However, for short waves (the period $T = 0.7$ sec corresponds to a wavelength of 0.765 m), there is a considerable discrepancy. We should note that for short waves, different sources give different results. The experimental value of the maximum dimensionless amplitude of flexural stresses is approximately 0.5, and its numerical values are 1.2 in [1], 0.38 in [4], and 0.78 in the present paper. For long waves, almost all numerical results of [1, 4] are in good agreement with the experiment.

Long-Wave Approximation. We consider the boundary-value problem (5)–(12) subject to the following conditions: $k \ll 1$, $\beta = O(1)$, $H_0 = O(1)$, $\gamma = O(k^2)$, and $d/a = o(1)$. In the basic approximation, we assume that the plate has a weak effect on the wave motion, if the wavelength of the incident wave is sufficiently long. Then, from Eq. (9) we can eliminate the term $\gamma\Phi(x, 0)$, which has order of magnitude $O(k^2)$ as $k \to 0$, and obtain the following solution of the reduced equation subject to boundary conditions (10):

$$W(x) \approx N[\exp(ikx) + k^2 W_1(x) + ik^3 W_2(x)], \tag{22}$$

where $N = (1 - \alpha + \beta k^4)^{-1}$. The functions $W_j(x)$ ($j = 1$ and 2) satisfy the equation

$$\beta \frac{d^4 W_j}{dx^4} + (1 - \alpha) W_j = 0 \qquad (|x| < 1) \tag{23}$$

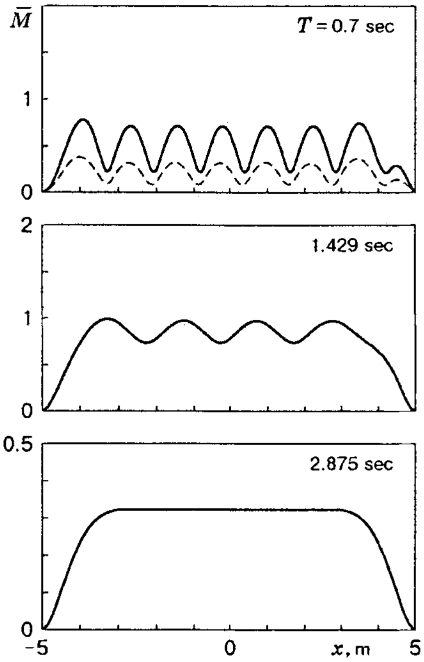

Fig. 2

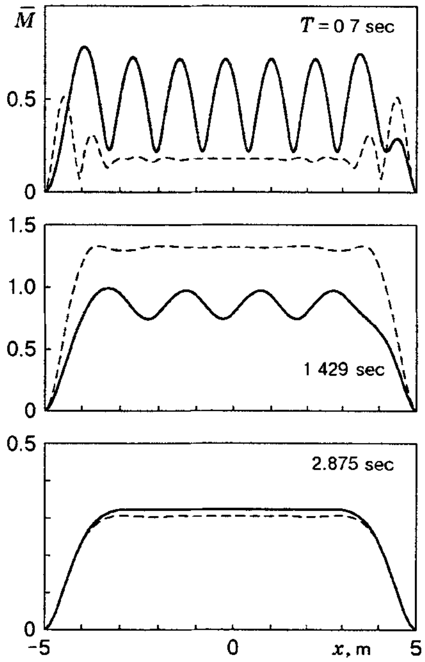

Fig. 3

and the boundary conditions

$$\frac{d^2W_1}{dx^2} = \exp(\pm ik), \qquad \frac{d^3W_1}{dx^3} = 0; \tag{24}$$

$$\frac{d^2W_2}{dx^2} = 0, \qquad \frac{d^3W_2}{dx^3} = \exp(\pm ik) \qquad (x = \pm 1). \tag{25}$$

It follows from the solutions of the boundary-value problems (23), (24) and (23), (25) that for small β, the functions $W_1(x)$ and $W_2(x)$ are localized in neighborhoods of the ends of the beam, and the sizes of these neighborhoods are $O(\beta^{1/4})$ in order of magnitude.

Substituting (22) into (8), we obtain a boundary-value problem for the correction to the velocity potential due to the presence of the plate. In general, approximation (22) is valid if this correction is small in absolute value for $|x| < 1$. Analysis of the long-wave approximation shows that this condition is satisfied if and only if $\alpha + \beta k^4 \ll 1$. This inequality takes into account that $W_1 = O(\beta^{1/2})$ and $W_2 = O(\beta^{3/4})$ as $\beta \to 0$.

We note that the long-wave approximation allows us to obtain solutions for the plate deflection and the stresses in analytical form. Figure 3 shows the dimensionless amplitudes of the bending moments obtained by the long-wave approximation (dashed curves) and the normal mode method (solid curves) for the same parameters of the incident wave as in Fig. 2. One can see that for short waves, the long-wave approximation gives an incorrect distribution of bending moments along the plate and their amplitudes. For incident waves of moderate wavelength, the long-wave approximation gives a simplified description. However for waves whose wavelength is comparable with the length of the plate, the approximate analytical solution is in good agreement with the numerical solution obtained within the framework of the full linear model.

The results of this paper show that the long-wave approximation can describe the hydroelastic behavior of floating plates for incident waves whose wavelength is greater than the total length of the plate.

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